

UNIFORM NULL CONTROLLABILITY FOR A DEGENERATING REACTION-DIFFUSION SYSTEM APPROXIMATING A SIMPLIFIED CARDIAC MODEL

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ABSTRACT. This paper is devoted to analyze the null controllability of a nonlinear reaction-diffusion system approximating a parabolic-elliptic system modeling electrical activity in the heart. The uniform, with respect to the degenerating parameter, null controllability of the approximating system by means of a single control is shown. The proof is based on the combination of Carlemans estimates and weighted energy inequalities.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and let ω and \mathcal{O} be (small) nonempty subsets of Ω , which will usually be referred as *control domains*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

In this paper we study the controllability and observability properties for a reaction-diffusion system which degenerates into a parabolic-elliptic system describing the cardiac electric activity in Ω ($\Omega \subset \mathbb{R}^3$ being the natural domain of the heart).

To state the model, we set $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ to represent, respectively, the spatial cellular and location $x \in \Omega$ of the *intracellular* and *extracellular* electric potentials, whose difference $v = v(t, x) = u_i - u_e$ is the *transmembrane* potential. The anisotropic properties of the two media are modeled by intracellular and extracellular conductivity tensors $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$. The surface capacitance of the membrane is represented by the constant $c_m > 0$. The transmembrane ionic current is represented by a nonlinear function $h(v)$ (the most interesting case being when the nonlinearity is cubic polynomial) depending on the value of the potential v .

The equations governing the cardiac electric activity are given by the coupled reaction-diffusion system:

$$(1.1) \quad \begin{aligned} c_m \partial_t v - \operatorname{div}(\mathbf{M}_i(x) \nabla u_i) + h(v) &= f 1_\omega && \text{in } Q, \\ c_m \partial_t v + \operatorname{div}(\mathbf{M}_e(x) \nabla u_e) + h(v) &= g 1_{\mathcal{O}} && \text{in } Q, \end{aligned}$$

where f and g are stimulation currents applied, respectively, to ω and \mathcal{O} .

System (1.1) is known as the *bidomain model*.

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We complete the bidomain model with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$(1.2) \quad u_i = 0 \text{ and } u_e = 0 \quad \text{on } \Sigma$$

and with initial data for the transmembrane potential

$$(1.3) \quad v(0, x) = v_0(x), \quad x \in \Omega.$$

It is important to point out that realistic models describing electrical activities include a system of ODE's for computing the ionic current as a function of the transmembrane potential and a serie of additional "gating variables" aiming to model the ionic transfer across the cell membrane (see [11, 17, 23, 24]).

Assume $f1_\omega = g1_\mathcal{O}$. If $\mathbf{M}_i = \mu\mathbf{M}_e$ for some constant $\mu \in \mathbb{R}$, then system (1.1) is equivalent to the parabolic-elliptic system:

$$(1.4) \quad \begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \operatorname{div} (\mathbf{M}_e(x) \nabla v) + h(v) = f1_\omega & \text{in } Q, \\ -\operatorname{div} (\mathbf{M}(x) \nabla u_e) = \operatorname{div} (\mathbf{M}_i(x) \nabla v) & \text{in } Q, \\ v = 0, \ u_e = 0 & \text{on } \Sigma, \\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

where $M = M_i + M_e$. System (1.4) is known as the *monodomain model*.

We approximate the monodomain model by the family of parabolic systems:

$$(1.5) \quad \begin{cases} c_m \partial_t v^\epsilon - \frac{\mu}{\mu+1} \operatorname{div} (\mathbf{M}_e(x) \nabla v^\epsilon) + h(v^\epsilon) = f^\epsilon 1_\omega & \text{in } Q, \\ \epsilon \partial_t u_e^\epsilon - \operatorname{div} (\mathbf{M}(x) \nabla u_e^\epsilon) = \operatorname{div} (\mathbf{M}_i(x) \nabla v^\epsilon) & \text{in } Q, \\ v^\epsilon = 0, \ u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0, \ u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega. \end{cases}$$

Since $v = u_i - u_e$ in the bidomain model, it is natural decompose the initial condition v_0 as $v_0 = u_{i,0} - u_{e,0}$.

The aim of this paper is to give an answer to the following question:

If, for each $\epsilon > 0$, there exists a control f^ϵ that drives the solution $(v^\epsilon, u_e^\epsilon)$ of (1.5) to zero at time $t = T$, i.e.

$$v^\epsilon(T) = u_e^\epsilon(T) = 0,$$

is it true that when $\epsilon \rightarrow 0$ the control sequence f^ϵ converges to a function f , that drives the solution (v, u_e) of (1.4) to zero at time $t = T$?

Since the bidomain model is a system of two coupled parabolic equations and the monodomain model is a system of parabolic-elliptic type, these two systems have, at least a priori, different control properties. Therefore, it is natural to ask if the controllability of the monodomain model can be seen as a limit process of the controllability of a family of parabolic systems.

It is worth to mention that systems of parabolic equations which degenerates into parabolic-elliptic ones arise in many areas, such as biology and models describing gravitational interaction of particles, see [4, 5, 20].

In order to answer the previous question, we consider the following linearized version of (1.5):

$$(1.6) \quad \begin{cases} c_m \partial_t v^\epsilon - \frac{\mu}{\mu+1} \operatorname{div} (\mathbf{M}_e(x) \nabla v^\epsilon) + a(t, x) v^\epsilon = f^\epsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\epsilon - \operatorname{div} (\mathbf{M}(x) \nabla u_e^\epsilon) = \operatorname{div} (\mathbf{M}_i(x) \nabla v^\epsilon) & \text{in } Q, \\ v^\epsilon = 0, \quad u_e^\epsilon = 0 & \text{on } \Sigma, \\ v^\epsilon(0) = v_0, \quad u_e^\epsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

where a is a bounded function.

Our objective then will be drive both v^ϵ and u^ϵ , solution of (1.6), to zero at time T by means of a control f^ϵ in such a way that the sequence of controls f^ϵ remains bounded when $\epsilon \rightarrow 0$. Accordingly, we consider the corresponding adjoint system:

$$(1.7) \quad \begin{cases} -c_m \partial_t \varphi^\epsilon - \frac{\mu}{\mu+1} \operatorname{div} (M_e(x) \nabla \varphi^\epsilon) + a(t, x) \varphi^\epsilon = \operatorname{div} (M_i(x) \nabla \varphi_e^\epsilon) & \text{in } Q, \\ -\varepsilon \partial_t \varphi_e^\epsilon - \operatorname{div} (M(x) \nabla \varphi_e^\epsilon) = 0 & \text{in } Q, \\ \varphi^\epsilon = 0, \quad \varphi_e^\epsilon = 0 & \text{on } \Sigma, \\ \varphi^\epsilon(T) = \varphi_T, \quad \varphi_e^\epsilon(T) = \varphi_{e,T} & \text{in } \Omega. \end{cases}$$

It is very easy to prove that our task turns out to be equivalent to the following observability inequality:

$$(1.8) \quad \epsilon \|\varphi_e^\epsilon(0)\|_{L^2(\Omega)}^2 + \|\varphi^\epsilon(0)\|_{L^2(\Omega)}^2 \leq C \iint_{Q_\omega} |\varphi^\epsilon|^2 dx dt, \quad Q_\omega := \omega \times (0, T),$$

for all $(\varphi_T, \varphi_{e,T}) \in L^2(\Omega)^2$ and a constant $C = C(\epsilon, \Omega, \omega, T)$ that remains bounded when $\epsilon \rightarrow 0$.

Let us now mention some works that have been devoted to the theoretical and numerical study of the bidomain model (1.1). In [6], it is proved the existence of weak solutions to (1.1) using the theory of evolution variational inequalities in Hilbert spaces. Applying the same approach, Sanfelici [26] proves the convergence of Galerkin approximations for this model. Recently, Bendahmane and Karlsen [1] proved the existence and uniqueness for a nonlinear version of the bidomain equations (1.1) by using a uniformly parabolic regularization of the system and the Faedo–Galerkin method.

Regarding finite volume (FV) schemes for cardiac problems, Bendahmane and Karlsen [2] analyse a FV method for the bidomain model, supplying various existence, uniqueness and convergence results. Bendahmane, Bürger and Ruiz [3] analyse a parabolic-elliptic system with Neumann boundary conditions, adapting the approach in [2]; they also provide numerical experiments.

Let us now recall some results on the controllability for systems of parabolic equations. In [15] the controllability of a quite general two coupled linear parabolic system is studied and null controllability is obtained by means of Carleman inequalities. In [19], using a different strategy, the controllability of a reaction-diffusion system of a simple two coupled parabolic equations is analyzed, the authors prove the null controllability for the linear system and the local null controllability of the nonlinear system. Another relevant work concerning to the controllability of coupled systems is [10], in which the authors analyze the null controllability of a cascade system of m coupled parabolic equations and the authors are able to obtain null controllability for the cascade system whenever they have a good coupling structure. But, unlike the present work, the aforementioned works are devoted to

systems that do not degenerate. Actually, if one follow their proofs, it can be seen that the constant C appearing in the observability inequality (1.8) is of order of ϵ^{-1} , which degenerates when $\epsilon \rightarrow 0$. Therefore, a careful analysis is required in order to guarantee uniform controllability with respect to the degenerating parameter ϵ . This will be done combining precise, with respect to ϵ , Carleman estimates and weighted energy inequalities. This kind of analysis has been used several times in the case of parabolic equations degenerating into hyperbolic ones (see [16, 7, 13]) and hyperbolic equations degenerating into parabolic ones (see [21, 22]) but, as far as we know, this is the first time that controllability of parabolic systems degenerating into parabolic-elliptic systems is studied.

Concerning to the controllability of the bidomain model, since we have couplings given by time derivatives of the electrical potentials on both equations, it is very difficult to analyze whether the bidomain model is null controllable or not, even with controls on both equations. As far as we know, this problem is still open. Regarding the monodomain model, since the solution to the parabolic equation enters as a source term in the elliptic equation, the following Theorem holds:

Theorem 1.1. *Given v_0 in $L^2(\Omega)$ and*

$$(1.9) \quad q_N \in (2, \infty) \text{ if } N = 1, 2, \quad \frac{N+2}{2} < q_N < 2\frac{N+2}{N-2} \text{ if } N \geq 3.$$

We have:

- *If h is $C^1(\mathbb{R})$, global lipschitz and satisfies $h(0) = 0$. Then there exists a control $f \in L^2(\omega \times (0, T))$ such that the solution (v, u_e) of (1.4) satisfies*

$$v(T) = u_e(T) = 0.$$

- *If h is of class C^1 satisfying*

$$(1.10) \quad h(0) = 0, \quad \frac{h(v_1) - h(v_2)}{v_1 - v_2} \geq -C, \quad \forall v_1 \neq v_2,$$

$$(1.11) \quad 0 < \liminf_{|v| \rightarrow \infty} \frac{h(v)}{v^3} \leq \limsup_{|v| \rightarrow \infty} \frac{h(v)}{v^3} < \infty,$$

and $v_0 \in H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}), q_N}(\Omega)$, with $\|v_0\|_{L^\infty} \leq \gamma$, for sufficient small γ . There exists a control $f \in L^{q_N}(\omega \times (0, T))$ such that the solution (v, u_e) of (1.4), with $(v, u_e) \in W_{q_N}^{2,1}(Q)^2$, satisfies

$$v(T) = u_e(T) = 0.$$

Proof of Theorem 1.1 can be obtained from a more general Theorem in [8].

For a general discussion about the controllability of parabolic systems see the survey paper [18].

In this paper we prove null controllability to (1.6) for each $\epsilon > 0$ by establishing an observability estimate like (1.8) for its dual system (1.7). Moreover, the estimate on the control we obtain are uniform with respect to ϵ , i.e., the constant C appearing in (1.8) does not depend on ϵ . In addition, we study the controllability of the nonlinear system (1.5) obtaining, under some assumptions on the nonlinearity and the initial data, uniform null controllability.

This paper is organized as follows: In Section 2, we state the main results. Section 3 is devoted to prove a Carleman inequality for the adjoint system (1.7).

In Section 4 we show the uniform null controllability for (1.6). Section 5 deals with the uniform null controllability for the nonlinear system (1.5).

2. MAIN RESULTS

Throughout this paper we will assume that the matrices M_j , $j = i, e$ are C^∞ , bounded, symmetric and positive semidefinite.

We have the following existence Theorem.

Theorem 2.1. *Under conditions (1.10) and (1.11). If $(v_0, u_{0,e}) \in L^2(\Omega)^2$ and $f \in L^2(Q)$, then system (1.5) have a unique weak solution $(v^\varepsilon, u_e^\varepsilon)$ and $(v^\varepsilon, u_e^\varepsilon) \in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; H_0^1(\Omega))$ such that $\partial_t v^\varepsilon$ and $\varepsilon \partial_t u_e^\varepsilon$ belong to $L^2(0, T, H^{-1}(\Omega)) + L^{4/3}(Q)$ and $L^2(0, T, H^{-1}(\Omega))$.*

The proof of Theorem 2.1 can be done exactly as in [1], so we omit it.

Our first main result is a uniform Carleman estimate for the adjoint system (1.7).

Theorem 2.2. *There exist positive constants $C = C(\Omega, \omega_0)$, λ_0 and s_0 so that for every $\varphi_T, \varphi_{e,T} \in L^2(\Omega)$, $a \in L^\infty(Q)$, the solution $(\varphi^\varepsilon, \varphi_e^\varepsilon)$ of (1.7) satisfies*

$$(2.1) \quad \begin{aligned} & \iint_Q e^{3s\alpha} |\rho^\varepsilon|^2 dxdt + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi^\varepsilon|^2 dxdt \\ & \leq C e^{6\lambda \|\psi\|} s^7 \lambda^4 \int_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi^\varepsilon|^2 dxdt, \end{aligned}$$

where $\rho^\varepsilon = \operatorname{div}(M(x) \nabla \varphi_e^\varepsilon(x, t))$, for every $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3} + \|a\|_{L^\infty}^{2/5})T^2 + T^4)s_0$, $\lambda \geq \lambda_0$ and appropriate weight functions ϕ and α defined in (3.3) and (3.4), respectively.

Proof of Theorem 2.2 is given in Section 3 .

Our second main result gives the null controllability of (1.6).

Theorem 2.3. *Given v_0 and $u_{e,0}$ in $L^2(\Omega)$. For each $\varepsilon > 0$, there exists a control $f^\varepsilon \in L^2(\omega \times (0, T))$ so that the associated solution to (1.6) is driven to zero at time T . That is, the associated solution satisfies*

$$v^\varepsilon(T) = 0, \quad u_e^\varepsilon(T) = 0.$$

Moreover, the control f^ε satisfies

$$(2.2) \quad \|f^\varepsilon 1_\omega\|_{L^2(Q)} \leq C(\|v_0\|_{L^2(\Omega)} + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}).$$

Theorem 2.3 is proved in Section 4.

The third main result of this paper is concerned with the uniform null controllability of the nonlinear parabolic system (1.5).

Theorem 2.4. *Given v_0 and $u_{e,0}$ in $L^2(\Omega)$ and let q_N satisfying (1.9). We have:*

- If h is $C^1(\mathbb{R})$, global lipschitz and satisfies $h(0) = 0$, there exists a control $f^\varepsilon \in L^2(\omega \times (0, T))$ such that the solution $(v^\varepsilon, u_e^\varepsilon)$ of (1.5) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Besides, the control f^ε has the estimate

$$(2.3) \quad \|f^\varepsilon 1_\omega\|_{L^2(Q)} \leq C(\|v_0\|_{L^2(\Omega)} + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}).$$

• Let h be a C^1 function satisfying (1.10) and (1.11) and the initial data $(v_0, u_{e,0}) \in (H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}), q_N}(\Omega))^2$, with $\|(v_0, u_{e,0})\|_{L^\infty} \leq \gamma$, for sufficient small γ does not depending on ε . There exists a control $f^\varepsilon \in L^{q_N}(\omega \times (0, T))$ such that the solution $(v^\varepsilon, u_e^\varepsilon)$ of (1.5), with $(v^\varepsilon, u_e^\varepsilon) \in (W_{q_N}^{2,1}(Q))^2$, satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

Moreover, the control f^ε has the estimate

$$(2.4) \quad \|f^\varepsilon 1_\omega\|_{L^{q_N}(Q)}^2 \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2).$$

We prove Theorem 2.4 in Section 5.

3. A CARLEMAN TYPE INEQUALITY

This section is devoted to prove Theorem 2.2.

To simplify the notation, we drop the index ε and, since the only constant which matters to us is ε , we will suppose that all other constants are equal to one. The adjoint system (1.7) then writes as

$$(3.1) \quad \begin{cases} -\partial_t \varphi - \operatorname{div}(M_e(x) \nabla \varphi) + a(x, t) \varphi = \operatorname{div}(M_i(x) \nabla \varphi_e) & \text{in } Q, \\ -\varepsilon \partial_t \varphi_e - \operatorname{div}(M(x) \nabla \varphi_e) = 0 & \text{in } Q, \\ \varphi = 0, \quad \varphi_e = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T, \quad \varphi_e(T) = \varphi_{e,T} & \text{in } \Omega. \end{cases}$$

Suppose that φ_T and $\varphi_{e,T}$ are regular enough. Taking $\rho(x, t) = \operatorname{div}(M_i(x) \nabla \varphi_e(x, t))$, we see that the pair (φ, ρ) satisfies

$$(3.2) \quad \begin{cases} -\partial_t \varphi - \operatorname{div}(M_e(x) \nabla \varphi) + a(x, t) \varphi = \rho & \text{in } Q, \\ -\varepsilon \partial_t \rho - \operatorname{div}(M(x) \nabla \rho) = 0 & \text{in } Q, \\ \varphi = 0, \quad \rho = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi_T, \quad \rho(T) = \rho_T & \text{in } \Omega. \end{cases}$$

Before start proving the Carleman inequality (2.1), let us define several weight functions which will be usefull in the sequel.

Lemma 3.1. *Let ω_0 be an arbitrary nonempty open set such that $\overline{\omega_0} \subset \omega \subset \Omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$\psi(x) > 0, \forall x \in \Omega, \quad \psi \equiv 0 \text{ on } \partial\Omega, \quad |\nabla \psi(x)| > 0 \quad \forall x \in \overline{\Omega \setminus \omega_0}.$$

Proof. The proof of this lemma is given in [9]. □

From Lemma 3.1 we introduce the weight functions

$$(3.3) \quad \phi(x, t) = \frac{e^{\lambda(\psi(x) + m\|\psi\|)}}{t(T-t)}; \quad \phi^*(t) = \min_{x \in \overline{\Omega}} \phi(x, t) = \frac{e^{\lambda m\|\psi\|}}{t(T-t)};$$

$$(3.4) \quad \alpha(x, t) = \frac{e^{\lambda(\psi(x) + m\|\psi\|)} - e^{2\lambda m\|\psi\|}}{t(T-t)}; \quad \alpha^*(t) = \max_{x \in \overline{\Omega}} \alpha(x, t) = \frac{e^{\lambda(m+1)\|\psi\|} - e^{2\lambda m\|\psi\|}}{t(T-t)},$$

for a parameter $\lambda > 0$ and a constant $m > 1$. Here

$$\|\psi(x)\| = \max_{x \in \overline{\Omega}} |\psi(x)|.$$

Remark 1. From the definition of α and α^* we have that $3\alpha^* \leq 2\alpha$ (for λ large enough!). Moreover

$$\phi^*(t) \leq \phi(x, t) \leq e^{\lambda\|\psi\|} \phi^*(x, t)$$

and

$$|\partial_t \alpha^*| \leq e^{2\lambda\|\psi\|} T \phi^2.$$

Now we prove Theorem 2.2.

Proof of Theorem 2.2 . For an easier comprehension, we divide the proof in several steps::

• **Step 1** *First estimate for the parabolic system*

In this step we obtain a first Carleman estimate for the adjoint system.

We consider a set ω_1 such that $\omega_0 \subset \subset \omega_1 \subset \subset \omega$ and apply Carleman inequalities (6.2), with $\varepsilon = 1$, and (6.15) to φ and ρ , respectively, to obtain

$$\begin{aligned} & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\varphi_t|^2 dxdt + s^{-1} \iint_Q \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\ (3.5) \quad & + s^3 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |\varphi|^2 dxdt + s \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla \varphi|^2 dxdt \\ & \leq C \left(\iint_Q e^{2s\alpha} (|\rho|^2 + |\varphi|^2) dxdt + s^3 \lambda^4 \iint_{Q_{\omega_1}} \phi^3 e^{2s\alpha} |\varphi|^2 dxdt \right), \end{aligned}$$

and

$$\begin{aligned} & \iint_Q s^{-1} e^{2s\alpha} |\partial_t \rho|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \rho \right|^2 dxdt \\ (3.6) \quad & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dxdt \\ & \leq C e^{\lambda\|\psi\|} s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_{\omega_1}} \phi^4 e^{2s\alpha} |\rho|^2 dxdt. \end{aligned}$$

Next, we add (3.5) and (3.6) and absorb the lower order terms in the right-hand side, we get this way

(3.7)

$$\begin{aligned}
& \iint_Q \phi^{-1} e^{2s\alpha} |\varphi_t|^2 dxdt + \iint_Q \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\
& + s^4 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |\varphi|^2 dxdt + s^2 \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla \varphi|^2 dxdt \\
& + \varepsilon^2 \iint_Q e^{2s\alpha} |\partial_t \rho|^2 dxdt + \iint_Q e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \rho \right|^2 dxdt \\
& + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dxdt \\
& \leq C \left(e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_{\omega_1}} \phi^4 e^{2s\alpha} |\rho|^2 dxdt + s^4 \lambda^4 \iint_{Q_{\omega_1}} \phi^3 e^{2s\alpha} |\varphi|^2 dxdt \right).
\end{aligned}$$

for $s \geq (T + (1 + \|a\|_{L^\infty}^{2/3} + \|a\|_{L^\infty}^{2/5})T^2 + T^4)s_0$.

At this point a remark has to be done. If we were trying to control (1.6) with controls on both equations, inequality (3.7) would be sufficient.

• **Step 2** *Estimation of the local integral of ρ .*

In this step we estimate the local integral involving ρ in the right-hand side of (3.7). It will be done using equation (3.2)₁. Indeed, we consider a function ξ satisfying

$$\xi \in C_0^\infty(\omega), \quad 0 \leq \xi \leq 1, \quad \xi(x) = 1 \quad \forall x \in \omega_1$$

and then

$$\begin{aligned}
& C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_{\omega_1}} e^{2s\alpha} \phi^4 |\rho|^2 dxdt \\
& \leq C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 |\rho|^2 \xi dxdt \\
& = C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 \rho (-\varphi_t - \operatorname{div}(M_e \nabla \varphi) + a\varphi) \xi dxdt \\
& := E + F + G,
\end{aligned}$$

In the sequel we estimate each parcel in the expression above.

$$\begin{aligned}
E &= C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} s \partial_t \alpha e^{2s\alpha} \phi^4 \rho \varphi \xi dxdt \\
&+ C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^3 \phi_t \rho \varphi \xi dxdt + C e^{\lambda \|\psi\|} s^4 \lambda^4 \iint_{Q_\omega} e^{2s\alpha} \phi^4 \partial_t \rho \varphi \xi dxdt \\
&:= E_1 + E_2 + E_3.
\end{aligned}$$

It is immediate to see that

(3.8)

$$E_1 + E_2 \leq \frac{1}{10}s^4\lambda^4 \iint_{Q_\omega} e^{2s\alpha}\phi^4|\rho|^2 dxdt + Ce^{2\lambda\|\psi\|}s^8\lambda^4 \iint_{Q_\omega} e^{2s\alpha}\phi^8|\varphi|^2 dxdt$$

and

$$E_3 \leq \frac{\varepsilon^2}{2} \iint_{Q_\omega} e^{2s\alpha}|\partial_t \rho|^2 dxdt + Ce^{2\lambda\|\psi\|}\varepsilon^{-2}s^8\lambda^8 \iint_{Q_\omega} e^{2s\alpha}\phi^8|\varphi|^2 dxdt.$$

Next,

$$\begin{aligned} e^{-\lambda\|\psi\|}s^{-4}\lambda^{-4}F &= \sum_{i,j=1}^N \iint_{Q_\omega} s\partial_{x_i}\alpha e^{2s\alpha}\phi^4\rho(M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha}\phi^3\partial_{x_i}\phi\rho(M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha}\phi^4\partial_{x_i}\rho(M_e^{ij}\partial_{x_j}\varphi)\xi dxdt \\ &\quad + \sum_{i,j=1}^N \iint_{Q_\omega} e^{2s\alpha}\phi^4\rho(M_e^{ij}\partial_{x_j}\varphi)\partial_{x_i}\xi dxdt \end{aligned}$$

and we can show that

$$\begin{aligned} F &\leq \frac{1}{10}s^4\lambda^4 \iint_{Q_\omega} e^{2s\alpha}\phi^4|\rho|^2 dxdt + \frac{1}{6}s^2\lambda^2 \iint_{Q_\omega} e^{2s\alpha}\phi^2|\nabla\rho|^2 dxdt \\ &\quad + Ce^{2\lambda\|\psi\|}s^8\lambda^8 \iint_{Q_\omega} e^{2s\alpha}\phi^8|\varphi|^2 dxdt + \frac{1}{2} \iint_{Q_\omega} e^{2s\alpha}|\partial_{x_i x_j}^2 \rho|^2 dxdt. \end{aligned}$$

Finally,

$$\begin{aligned} G &\leq \frac{1}{10}s^4\lambda^4 \iint_{Q_\omega} e^{2s\alpha}\phi^4|\rho|^2 dxdt \\ &\quad + Ce^{2\lambda\|\psi\|}s^4\lambda^4|a|_{L^\infty}^2 \iint_{Q_\omega} e^{2s\alpha}\phi^4|\varphi|^2 dxdt. \end{aligned}$$

Putting E , F and G together in (3.7), we get

(3.9)

$$\begin{aligned}
& \iint_Q e^{2s\alpha} |\varphi_t|^2 dxdt + \iint_Q e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\
& + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\varphi|^2 dxdt + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \varphi|^2 dxdt \varepsilon^2 \iint_Q e^{2s\alpha} |\partial_t \rho|^2 dxdt \\
& + \iint_Q e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \rho \right|^2 dxdt + s^4 \lambda^4 \iint_Q \phi^4 e^{2s\alpha} |\rho|^2 dxdt \\
& + s^2 \lambda^2 \iint_Q \phi^2 e^{2s\alpha} |\nabla \rho|^2 dxdt \\
& \leq C e^{2\lambda \|\psi\|} \varepsilon^{-2} s^8 \lambda^8 \iint_{Q_\omega} e^{2s\alpha} \phi^8 |\varphi|^2 dxdt.
\end{aligned}$$

Using (3.9) we could prove that, for every $\varepsilon > 0$, system (1.6) is null controllable, but the sequence of control obtained this way will not be bounded when $\varepsilon \rightarrow 0$. Therefore, we need to go further and improve (3.9). This will be done in the next step.

• **Step 3. An energy Inequality.**

The reason why we do not obtain a bounded sequence of controls out of step 2 is because of the term ε^{-2} in the right-hand side of (3.9). In this step we prove a weighted energy inequality for equation (3.2)₂, which will be used to, somehow, compensate this ε^{-2} term.

Let us introduce the function

$$y = e^{\frac{3}{2}s\alpha^*} \rho,$$

which solves the system

$$(3.10) \quad \begin{cases} \varepsilon \partial_t y - \operatorname{div}(M(x) \nabla y) = \varepsilon \frac{3}{2} s \partial_t \alpha^* e^{\frac{3}{2}s\alpha^*} \rho & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y(T) = 0 & \text{in } \Omega. \end{cases}$$

We multiply (3.10) by y and integrate on Ω , we get

$$\frac{\varepsilon}{2} \frac{d}{dt} \|y(t)\|_{L^2(\Omega)}^2 + C \|\nabla y(t)\|_{L^2(\Omega)}^2 \leq \varepsilon \frac{3}{2} \int_{\Omega} s \partial_t \alpha^*(t) e^{\frac{3}{2}s\alpha^*(t)} \rho(t) y(t) dx.$$

Integrating this last inequality from 0 to T and using Poincaré's and Young's inequalities, it is not difficult to see that

$$(3.11) \quad \iint_Q e^{3s\alpha^*} |\rho|^2 dxdt \leq C \varepsilon^2 e^{4\lambda \|\psi\|} \iint_Q s^3 \phi^4 e^{2s\alpha} |\rho|^2 dxdt.$$

From (3.9) and (3.11) we obtain

$$(3.12) \quad \iint_Q e^{3s\alpha^*} |\rho|^2 dxdt \leq C e^{6\lambda \|\psi\|} s^7 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dxdt.$$

This estimate gives a global estimate of ρ in terms of a local integral of ϕ , with a bounded constant.

• **Step 4. Last estimates and conclusion.**

In order to finish the prove of Theorem 2.2, we combine inequality (3.12) and an another Carleman inequality to equation (3.2)₁. Indeed, we have

$$\begin{aligned}
 (3.13) \quad & \iint_Q s^{-1} \phi^{-1} e^{3s\alpha} |\varphi_t|^2 dxdt + s^{-1} \iint_Q \phi^{-1} e^{3s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 \varphi \right|^2 dxdt \\
 & + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi|^2 dxdt + s \lambda^2 \iint_Q \phi e^{3s\alpha} |\nabla \varphi|^2 dxdt \\
 & \leq C \left(\iint_Q e^{3s\alpha} |\rho|^2 dxdt + s^3 \lambda^4 \iint_{Q_\omega} \phi^3 e^{3s\alpha} |\varphi|^2 dxdt \right),
 \end{aligned}$$

where φ is, together with ρ , solution of (3.2).

Here, we just changed the weight $e^{2s\alpha}$ by $e^{3s\alpha}$. The proof of (3.13) is the same as the one given by Theorem 6.1, just taking a slightly different change of variable in (6.3).

Next, since $e^{3s\alpha} \leq e^{3s\alpha^*}$, we have

$$\iint_Q e^{3s\alpha} |\rho|^2 dxdt \leq \iint_Q e^{3s\alpha^*} |\rho|^2 dxdt$$

and by (3.12),

$$\iint_Q e^{3s\alpha} |\rho|^2 dxdt \leq C e^{6\lambda \|\psi\|} s^7 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dxdt.$$

From (3.12) and (3.13), it follows that

$$(3.14) \quad \iint_Q e^{3s\alpha} |\rho|^2 dxdt + s^3 \lambda^4 \iint_Q \phi^3 e^{3s\alpha} |\varphi|^2 dxdt \leq C e^{6\lambda \|\psi\|} s^7 \lambda^4 \iint_{Q_\omega} \phi^8 e^{2s\alpha} |\varphi|^2 dxdt,$$

which is exactly (2.1).

By density, we can show that (3.14) remains true when we consider initial data in $L^2(\Omega)$. Therefore, the Carleman inequality (2.1) holds for all initial data in $L^2(\Omega)$.

This finishes the proof of Theorem 2.2. \square

4. NULL CONTROLLABILITY FOR THE LINEARIZED SYSTEM

This section is intended to prove the null controllability of linearized equation (1.6). It will be done by showing observability inequality (1.8) for the adjoint system (1.7) and solving a minimization problem. The arguments used here are classical in control theory for linear PDE's, so that we just give a sketch of the proof.

• *Sketch of the proof of Theorem 2.3.* By standard energy inequality for system (3.2), we can prove, using (3.14), that

$$(4.1) \quad \epsilon \|\rho(0)\|_{L^2(\Omega)}^2 + \|\varphi(0)\|_{L^2(\Omega)}^2 \leq C_1 e^{C_2 \|a\|_\infty T} \iint_{Q_\omega} |\varphi|^2 dxdt,$$

for some constants $C_1, C_2 > 0$.

Since $\rho(x, t) = \operatorname{div}(M(x) \nabla \varphi_e(x, t))$ and $\varphi_e = 0$ on $\partial\Omega$, we have

$$\|\varphi_e(t)\|_{H^2(\Omega)} \leq C \|\rho(t)\|_{L^2(\Omega)},$$

for all $t \in [0, T]$. Therefore, it follows from (4.1) that

$$(4.2) \quad \epsilon \|\varphi_e(0)\|_{L^2(\Omega)}^2 + \|\varphi(0)\|_{L^2(\Omega)}^2 \leq C \iint_{Q_\omega} |\varphi|^2 dx dt,$$

which is the observability inequality (1.8).

From (4.2) and the density of smooth solutions in the space of solutions of (3.1) with initial data in $L^2(\Omega)$, we see that the above observability inequality is satisfied by all solutions of (1.7) with initial data in $L^2(\Omega)$.

Now, in order to obtain the null controllability for linear system (1.6), we need to solve the following minimization problem:

Given φ_T and $\varphi_{e,T}$ in $L^2(\Omega)$,

$$(4.3) \quad \begin{aligned} & \text{Minimize } \mathcal{J}_\delta(\varphi_T, \varphi_{e,T}), \text{ with} \\ & \mathcal{J}_\delta(\varphi_T, \varphi_{e,T}) = \left\{ \frac{1}{2} \int_0^T \int_\omega |\varphi^\varepsilon|^2 dx dt + \varepsilon(u_{e,0}, \varphi_e^\varepsilon(0)) \right. \\ & \quad \left. + (v_0, \varphi^\varepsilon(0)) + \delta(\|\varphi_T\|_{L^2(\Omega)} + \|\varphi_{e,T}\|_{L^2(\Omega)}) \right\}, \end{aligned}$$

where (φ, φ_e) is the solution of the adjoint problem (1.7) with initial data $(\varphi_T, \varphi_{e,T})$.

It is an easy matter to check that \mathcal{J}_δ is strictly convex and continuous. So, in order to guarantee the existence of a minimizer, the only thing remaining to prove is the coercivity of \mathcal{J}_δ .

Using the observability inequality (1.8) for the adjoint system (1.7), the coercivity of \mathcal{J}_δ is straightforward. Therefore, for each $\delta > 0$, there exists a unique minimizer $(\varphi_{e,T}^\delta, \varphi_T^\delta)$ of \mathcal{J}_δ . Let us denote by $\varphi^{\varepsilon,\delta}$ the corresponding solution to (1.7) associated to this minimizer. Taking $f^{\varepsilon,\delta} = \varphi^{\varepsilon,\delta} 1_\omega$ as a control for (1.6), the duality between (1.6) and (1.7) gives

$$(4.4) \quad \|v^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} + \|u_e^{\varepsilon,\delta}(T)\|_{L^2(\Omega)} \leq \delta,$$

where $(v^{\varepsilon,\delta}, u_e^{\varepsilon,\delta})$ is the solution associated to the control $f^{\varepsilon,\delta}$. It also gives

$$(4.5) \quad \|f^{\varepsilon,\delta} 1_\omega\|_{L^2(Q)} \leq C(\|v_0\|_{L^2(\Omega)} + \varepsilon\|u_{e,0}\|_{L^2(\Omega)}).$$

From (4.4) and (4.5), we get a control f^ε (the weak limit of a subsequence of $f^{\varepsilon,\delta} 1_\omega$ in $L^2(\omega \times (0, T))$) that drives the solution of (1.6) to zero at time T . From (4.5) we have the following estimate on the control f^ε ,

$$(4.6) \quad \|f^\varepsilon 1_\omega\|_{L^2(Q)} \leq C(\|v_0^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|u_{e,0}^\varepsilon\|_{L^2(\Omega)}).$$

This finishes the proof of Theorem 2.3. \square

5. THE NONLINEAR SYSTEM

In this section we prove Theorem 2.4. The proof is done applying fixed point arguments.

Proof of Theorem 2.4 (case 1): We consider the following linearization of system (1.5):

$$(5.1) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div} (\mathbf{M}_e(x) \nabla v^\varepsilon) + g(z) v^\varepsilon = f^\varepsilon 1_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div} (\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div} (\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

where

$$(5.2) \quad g(s) = \begin{cases} \frac{h(s)}{s} & \text{if } |s| > 0, \\ h'(s) & \text{if } s = 0. \end{cases}$$

Follows from Theorem 2.3 that for each $v_0, u_{e,0} \in L^2(\Omega)$ and $z \in L^2(Q)$, there exist a control function $f^\varepsilon \chi_\omega \in L^2(Q)$ such that the solution of (5.1) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

As we said before, the idea is to use a fixed point argument. For that, we will use the following generalized version of Kakutani's fixed point theorem, due to Glicksberg [14].

Theorem 5.1. *Let B be a non-empty convex, compact subset of a locally convex topological vector space X . If $\Lambda : B \rightarrow X$ is a set-valued mapping convex, compact and with closed graph. Then the set of fixed points of Λ is non-empty and compact.*

In order to apply Glicksberg's Theorem, we define a mapping $\Lambda : B \rightarrow X$ as follows

$$\Lambda(z) = \{v; (v, u_e) \text{ is a solution of (5.1), such that } v(T) = u_e(T) = 0, \\ \text{for a control } f \chi_\omega \text{ satisfying (2.2)}\}.$$

where $X = L^2(Q)$ and B is the ball

$$B = \{z \in L^2(0, T, H_0^1(\Omega)), \partial_t z \in L^2(0, T, H^{-1}(\Omega)); \\ \|z\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t z\|_{L^2(0, T; H^{-1}(\Omega))}^2 \leq M\}.$$

It is easy to see that Λ is well defined and that B is convex and compact in $L^2(Q)$. Now, we prove that Λ is convex, compact and has closed graphic. It will be done into the next steps.

- $\Lambda(B) \subset B$.

Let $z \in B$ and $v \in \Lambda(z)$. Since v satisfies (5.1)₁, the following inequality holds

$$(5.3) \quad \|v\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t v\|_{L^2(0, T; H^{-1}(\Omega))}^2 \leq K_1.$$

In this way, if $z \in B$ then $\Lambda(z) \subset B$, if we take $M = K_1$.

- $\Lambda(z)$ is closed in $L^2(Q)$.

Let $z \in B$ fixed, and $v_n \in \Lambda(z)$, such that $v_n \rightarrow v$. Let's prove that $v \in \Lambda(z)$. In fact, by definition we have that v_n is, together with a function $u_{e,n}$ and a control f_n , the solution of (5.1), with $\|f_n 1_\omega\|_{L^2(Q)} \leq C(\|v_0\|_{L^2(\Omega)} + \varepsilon \|u_{e,0}\|_{L^2(\Omega)})$. Therefore we can extract a subsequence of f_n , denoted by the same index, such that

$$f_n 1_\omega \rightarrow f 1_\omega \text{ weakly in } L^2(Q).$$

Since f_n is bounded, we can argue as in the previous section in order to obtain the inequality

$$(5.4) \quad \|v_n\|_{L^2(0,T;H_0^1(\Omega))}^2 + \|\partial_t v_n\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leq C$$

and follows that

$$\begin{cases} v_n \rightarrow v \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ v_n \rightarrow v \text{ strongly in } L^2(Q), \\ \partial_t v_n \rightarrow \partial_t v \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{cases}$$

Using the converges above and (5.1)₂, we see that there exists a function u_e such that

$$\begin{cases} u_{e,n} \rightarrow u_e \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ u_{e,n} \rightarrow u_e \text{ strongly in } L^2(Q), \\ \partial_t u_{e,n} \rightarrow \partial_t u_e \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{cases}$$

It is immediate that (u_e, v) is a controlled solution of (5.1) associated to the control f . Hence $v \in \Lambda(z)$ and $\Lambda(z)$ is closed and compact of $L^2(Q)$.

- Λ has closed graph in $L^2(Q) \times L^2(Q)$.

We need to prove that if $z_n \rightarrow z$, $v_n \rightarrow v$ strongly in $L^2(Q)$ and $v_n \in \Lambda(z_n)$, then $v \in \Lambda(z)$. Using previous steps, it is straightforward that $v \in \Lambda(z)$.

Therefore, we can apply Glicksberg Theorem to conclude that Λ has a fixed point. This proves Theorem 2.4 in the case which the nonlinearity is a C^1 global Lipschitz function.

Proof of Theorem 2.4 (case 2): The proof of the local null controllability in the case 2 is done exactly as in the equivalent one in [19].

We consider the linearization

$$(5.5) \quad \begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(\mathbf{M}_e(x) \nabla v^\varepsilon) + a(z) v^\varepsilon = f^\varepsilon \chi_\omega & \text{in } Q, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(\mathbf{M}(x) \nabla u_e^\varepsilon) = \operatorname{div}(\mathbf{M}_i(x) \nabla v^\varepsilon) & \text{in } Q, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0) = v_0, \quad u_e^\varepsilon(0) = u_{e,0} & \text{in } \Omega, \end{cases}$$

with $(v_0, u_{e,0}) \in (H_0^1(\Omega) \cap W^{2(1-\frac{1}{q_N}), q_N}(\Omega))^2$, $z \in L^\infty(Q)$ and

$$a(z) = \int_0^1 \frac{dh}{dz}(sz) ds.$$

It is not difficult to show the null controllability of (1.5) with a control in $L^2(\omega \times (0, T))$, but these kind of controls are not sufficient to use fixed point arguments in order to control the nonlinear system (1.4). Our strategy then will be to change a bit the functional (4.3) in order to get controls in $L^{q_N}(Q)$ and then apply a fixed point argument. In fact, we consider the functional:

$$(5.6) \quad \begin{aligned} & \text{Minimize } \mathcal{J}_\delta(\varphi_T, \varphi_{eT}), \text{ with} \\ & \mathcal{J}_\delta(\varphi_T, \varphi_{eT}) = \left\{ \frac{1}{2} \int_0^T \int_\omega e^{2s\alpha} \phi^8 |\varphi^\varepsilon|^2 dx dt + \varepsilon(u_{e,0} \varphi_e^\varepsilon(0)) \right. \\ & \quad \left. + (v_0, \varphi^\varepsilon(0)) + \delta \left(\|\varphi_T\|_{L^2(\Omega)} + \|\varphi_{e,T}\|_{L^2(\Omega)} \right) \right\}, \end{aligned}$$

where $(\varphi^\varepsilon, \varphi_e^\varepsilon)$ is the solution of the adjoint system (1.7) with initial data $(\varphi_T, \varphi_{e,T})$.

As before, it can be proved that (5.6) has a unique minimizer $(\varphi^{\varepsilon,\delta}, \varphi_e^{\varepsilon,\delta})$. Defining $f^{\varepsilon,\delta} = e^{2s\alpha} \phi^8 \varphi^{\varepsilon,\delta}$ and using the fact that $\varphi^{\varepsilon,\delta}$ is, together with a $\varphi_e^{\varepsilon,\delta}$, the solution of (1.7), we see that $f^{\varepsilon,\delta}$ is a solution of a heat equation with null initial data and right-hand side in $L^2(Q)$. Using the regularizing effect of the heat equation and arguing exactly as in [19] we can show the estimate

$$(5.7) \quad \|f^{\varepsilon,\delta} 1_\omega\|_{L^q(Q)} \leq C(\|v_0\|_{L^2(\Omega)}^2 + \varepsilon \|u_{e,0}\|_{L^2(\Omega)}^2).$$

Taking the limit when $\delta \rightarrow 0$, we get a control $f^\varepsilon 1_\omega \in L^q(Q)$ such that the associated solution $(v^\varepsilon, u_e^\varepsilon)$ to (5.5) satisfies

$$v^\varepsilon(T) = u_e^\varepsilon(T) = 0.$$

The proof is finished by applying Kakutani's fixed point Theorem for system (5.5), exactly as done in Theorem 6 in [19].

6. APPENDIX: SOME TECHNICAL RESULTS

Let $g \in L^2(Q)$ and $v_T \in L^2(\Omega)$. In this section we will consider the equation

$$(6.1) \quad \begin{cases} -\partial_t v(x, t) - \sum_{i,j=1}^N \partial_{x_i}(a_{ij}(x) \partial_{x_j} v(t, x)) = g(x, t) & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(T) = v_T & \text{in } \Omega, \end{cases}$$

under the assumption that the matrix a_{ij} has the form:

$$a_{ij} = \frac{M_{ij}}{\varepsilon},$$

where $(M_{ij})_{ij}$ is an elliptic matrix, i.e., there exists $\beta > 0$ such that $\sum_{i,j}^N M_{ij} \xi_j \xi_i \geq \beta |\xi|^2$ for all $\xi \in \mathbb{R}^N$.

6.1. A degenerating Carleman estimate.

Theorem 6.1. *There exists $\lambda_0 \geq 1$ and s_0 such that, for each, $\lambda > \lambda_0$ and $s \geq s_0(T + T^2 + T^4)$ the solution v of the equation (6.1) satisfies*

$$(6.2) \quad \begin{aligned} & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t v|^2 dx dt + s^{-1} \varepsilon^{-2} \iint_Q \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 v \right|^2 dx dt \\ & + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 e^{2s\alpha} |v|^2 dx dt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi e^{2s\alpha} |\nabla v|^2 dx dt \\ & \leq C \left(\iint_Q e^{2s\alpha} |g|^2 dx dt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 e^{2s\alpha} |v|^2 dx dt \right), \end{aligned}$$

with $C > 0$ depending on Ω , ω_0 , ψ and β .

Proof. For $s > 0$ and $\lambda > 0$ we consider the change of variable

$$(6.3) \quad w(t, w) = e^{s\alpha} v(t, w),$$

which implies

$$w(T, x) = w(0, x) = 0.$$

We have

$$(6.4) \quad L_1 w + L_2 w = g_s,$$

where

$$(6.5) \quad L_1 w = -\partial_t w + 2s\lambda \sum_{i,j=1}^N \phi a_{ij} \partial_{x_j} \psi \partial_{x_i} w + 2s\lambda^2 \sum_{i,j=1}^N \phi a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w,$$

$$(6.6) \quad L_2 w = -\sum_{i,j=1}^N \partial_{x_i} (a_{ij} \partial_{x_j} w) - s^2 \lambda^2 \sum_{i,j=1}^N \phi^2 a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w + s \partial_t \alpha w,$$

and

$$(6.7) \quad g_s = e^{s\alpha} g + s\lambda^2 \sum_{i,j=1}^N \phi a_{ij} \partial_{x_i} \psi \partial_{x_j} \psi w - s\lambda \sum_{i,j=1}^N \phi \partial_{x_i} (a_{ij} \partial_{x_j} \psi) w.$$

From (6.4),

$$(6.8) \quad \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 + 2(L_1 w, L_2 w)_{L^2(Q)} = \|g_s\|_{L^2(Q)}^2.$$

The idea is to analyze the terms appearing in $(L_1 w, L_2 w)_{L^2(Q)}$. First, we write

$$(L_1 w, L_2 w)_{L^2(Q)} = \sum_{i,j=1}^N I_{ij},$$

where I_{ij} is the inner product in $L^2(Q)$ of the i th term in the expression of $L_1 w$ and the j th term of $L_2 w$ and, after a long, but straightforward, calculation, we can show that

$$(6.9) \quad \begin{aligned} 2(L_1 w, L_2 w)_{L^2(Q)} &\geq 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_Q \phi^3 |\nabla \psi|^4 |w|^2 dx dt + 2s\lambda^2 \beta^2 \varepsilon^{-2} \iint_Q \phi |\nabla \psi|^2 |\nabla w|^2 dx dt \\ &\quad - C\varepsilon^{-2} \left(T^4 s^2 \lambda^4 + T s^2 \lambda^2 + T^2 s + s^3 \lambda^3 + T s^2 \lambda \right) \iint_Q \phi^3 |w|^2 dx dt \\ &\quad - C\varepsilon^{-2} (s\lambda + \lambda^2) \iint_Q \phi |\nabla w|^2 dx dt. \end{aligned}$$

We take $\lambda \geq \lambda_0$ and $s \geq s_0(T + T^2 + T^4)$, it follows from Remark 2 that

$$(6.10) \quad \begin{aligned} &2(L_1 w, L_2 w)_{L^2(Q)} + 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_{Q_{\omega_0}} \phi^3 |w|^2 dx dt \\ &\quad + 2s\lambda^2 \beta^2 \varepsilon^{-2} \iint_{Q_{\omega_0}} \phi |\nabla w|^2 dx dt \\ &\geq 2s^3 \lambda^4 \beta^2 \varepsilon^{-2} \iint_Q \phi^3 |w|^2 dx dt + 2s\lambda^2 \beta^2 \varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dx dt. \end{aligned}$$

Remark 2. Since $\overline{\Omega \setminus \omega_0}$ is compact and $|\nabla \psi| > 0$ on $\overline{\Omega \setminus \omega_0}$, there exists $\delta > 0$ such that

$$\beta |\nabla \psi| \geq \delta \text{ on } \overline{\Omega \setminus \omega_0}.$$

Putting (6.10) in (6.8), we get

$$\begin{aligned}
& \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 + 2\beta^{-2}s^3\lambda^4\delta^4\varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt \\
& \quad + 2s\lambda^2\delta^2\varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
(6.11) \quad & \leq \|g_s\|_{L^2(Q)}^2 + 2\beta^{-2}s^3\lambda^4\delta^4\varepsilon^{-2} \iint_{Q_{\omega_0}} \phi^3 |w|^2 dxdt \\
& \quad + 2s\lambda^2\delta^2\varepsilon^{-2} \iint_{Q_{\omega_0}} \phi |\nabla w|^2 dxdt.
\end{aligned}$$

Now we want to deal with the local integral involving ∇w in the right-hand side of (6.11). To this end we introduce a cut-off function ξ such that

$$\xi \in C_0^\infty(\omega), \quad 0 \leq \xi \leq 1, \quad \xi(x) = 1 \quad \forall x \in \omega_0.$$

Using the ellipticity condition on a_{ij} , we prove that

$$\begin{aligned}
& \beta\varepsilon^{-1} \iint_{Q_\omega} \phi \xi^2 |\nabla w|^2 dxdt \\
& \leq C \left(\iint_Q L_2 w \phi \xi^2 w dxdt + (sT + \varepsilon^{-1}s^2\lambda^2) \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right. \\
& \quad \left. + \lambda\varepsilon^{-1} \iint_{Q_\omega} \phi^{1/2} |\nabla w| \xi \phi^{1/2} w dxdt \right).
\end{aligned}$$

By Young's inequality we have

$$\begin{aligned}
& s\lambda^2\delta^2\varepsilon^{-2} \iint_{Q_\omega} \phi \xi^2 |\nabla w|^2 dxdt \\
& \leq \frac{1}{4} \iint_Q |L_2 w|^2 dxdt + C\beta^{-2}s^3\lambda^4(\delta^4 + \delta^2)\varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt.
\end{aligned}$$

Thus, inequality (6.11) gives

$$\begin{aligned}
& \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q_T)}^2 + \beta^{-2}s^3\lambda^4\varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt \\
(6.12) \quad & \quad + s\lambda^2\varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
& \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + \beta^{-2}s^3\lambda^4\varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).
\end{aligned}$$

Now we use the first two terms in left-hand side of (6.12) in order to add the integrals of $|\Delta w|^2$ and $|w_t|^2$ to the left-hand side of (6.12). This can be made using the expressions of $L_1 w$ and $L_2 w$. Indeed, from (6.5) and (6.6), we have

$$\begin{aligned}
(6.13) \quad & \iint_Q s^{-1} \phi^{-1} |\partial_t w|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} \phi^{-1} \sum_{i,j=1}^N |\partial_{x_i} (M_{ij} \partial_{x_j} w)|^2 dxdt \\
& + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
& \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).
\end{aligned}$$

Using the term in $|\partial_{x_i} (M_{ij} \partial_{x_j} w)|^2$ in the left-hand side of (6.13) and elliptic regularity, it is easy to show that

$$s^{-1} \varepsilon^{-2} \iint_Q \phi^{-1} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 w \right|^2 dxdt \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).$$

Estimate (6.13) then gives

$$\begin{aligned}
(6.14) \quad & \iint_Q s^{-1} \phi^{-1} |\partial_t w|^2 dxdt + s^{-1} \varepsilon^{-2} \iint_Q \phi^{-1} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 w \right|^2 dxdt \\
& + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 |w|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi |\nabla w|^2 dxdt \\
& \leq C \left(\|e^{s\alpha} g\|_{L^2(Q)}^2 + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 |w|^2 dxdt \right).
\end{aligned}$$

From (6.14) and the fact that $w = e^{s\alpha} v$, it is not difficult to finish the proof of Theorem 6.1. \square

6.2. A Slightly changed Carleman inequality.

Theorem 6.2. *There exists $\lambda_0 \geq 1$ and s_0 such that, for each, $\lambda > \lambda_0$ and $s \geq s_0(T + T^2 + T^4)$ the solution v of the equation (6.1) satisfies*

$$\begin{aligned}
(6.15) \quad & \iint_Q s^{-1} e^{2s\alpha} |\partial_t v|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 v \right|^2 dxdt \\
& + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^4 e^{2s\alpha} |v|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi^2 e^{2s\alpha} |\nabla v|^2 dxdt \\
& \leq C e^{\lambda \|\psi\|} \left(\iint_Q \phi e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^4 e^{2s\alpha} |v|^2 dxdt \right),
\end{aligned}$$

with $C > 0$ depending on Ω , ω_0 , ψ and β .

Proof. The starting point is to apply the Carleman inequality given in Theorem 6.1 for v , i.e.,

$$\begin{aligned}
(6.16) \quad & \varepsilon^2 \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |v_t|^2 dxdt + \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 v \right|^2 dxdt \\
& + s^3 \lambda^4 \iint_Q \phi^3 e^{2s\alpha} |v|^2 dxdt + s \lambda^2 \iint_Q \phi e^{2s\alpha} |\nabla v|^2 dxdt \\
& \leq C \left(\iint_Q e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \iint_{Q_\omega} \phi^3 e^{2s\alpha} |v|^2 dxdt \right).
\end{aligned}$$

Next, we introduce the function $y(x, t) = v(x, t)(\phi^*(t))^{\frac{1}{2}}$, which solves the system

$$(6.17) \quad \begin{cases} \varepsilon \partial_t y - \operatorname{div}(M(x) \nabla y) = -\varepsilon \frac{(T-2t)}{2} \phi^* y + (\phi^*(t))^{\frac{1}{2}} g & \text{in } Q, \\ y = 0 & \text{on } \Sigma. \end{cases}$$

Applying again the Carleman inequality given by Theorem 6.1, at this time for y , we obtain, for s large enough, that

$$\begin{aligned}
(6.18) \quad & \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t y|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 y \right|^2 dxdt \\
& + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 e^{2s\alpha} |y|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi e^{2s\alpha} |\nabla y|^2 dxdt \\
& \leq C \left(\iint_Q \phi^* e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 e^{2s\alpha} |y|^2 dxdt \right).
\end{aligned}$$

From the definition of y it is easy to show that

$$\iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |v_t(\phi^*)^{\frac{1}{2}}|^2 dxdt \leq \iint_Q s^{-1} \phi^{-1} e^{2s\alpha} |\partial_t y|^2 dxdt + \iint_Q e^{2s\alpha} \phi |y|^2 dxdt$$

and inequality (6.18) becomes

$$\begin{aligned}
(6.19) \quad & \iint_Q s^{-1} \phi^{-1} \phi^* e^{2s\alpha} |v_t|^2 dxdt + \varepsilon^{-2} \iint_Q s^{-1} \phi^* \phi^{-1} e^{2s\alpha} \left| \sum_{i,j=1}^N \partial_{x_i x_j}^2 v \right|^2 dxdt \\
& + s^3 \lambda^4 \varepsilon^{-2} \iint_Q \phi^3 \phi^* e^{2s\alpha} |v|^2 dxdt + s \lambda^2 \varepsilon^{-2} \iint_Q \phi \phi^* e^{2s\alpha} |\nabla v|^2 dxdt \\
& \leq C \left(\iint_Q \phi^* e^{2s\alpha} |g|^2 dxdt + s^3 \lambda^4 \varepsilon^{-2} \iint_{Q_\omega} \phi^3 \phi^* e^{2s\alpha} |v|^2 dxdt \right).
\end{aligned}$$

From Remark 1 the result follows. \square

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